Lifted Euclidean inequalities for the Integer Single Node Flow set with upper bounds

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Abstract

In this paper we discuss the polyhedral structure of the integer single node flow set with two possible values for the upper bounds on the arc flows. Such mixed integer sets arise as substructures in complex mixed integer programs for real application problems.

This work builds on results for the integer single node flow polytope with two arcs given by Agra and Constantino, 2006. Valid inequalities are extended to a new family, the *lifted Euclidean inequalities*, and a complete description of the convex hull is given. All the coefficients of the facet-defining inequalities can be computed in polynomial time.

We report on some computational experimentations for three problems: an inventory distribution problem, a facility location problem and a multi-item production planning model.

Keywords: valid inequalities; mixed integer programming; polyhedral description; single node flow set

1. Introduction

The description of the convex hull of elementary mixed integer sets has been useful in the generation of strong valid inequalities for general mixed integer problems. Particular cases of such elementary sets are the Single Node Flow (SNF) sets (see Figure 1):

$$\{(y, x) \in \mathbb{Z}_+^{|N|} \times \mathbb{R}^{|N|} : \sum_{t \in N} x_t \leq (=)(\geq) D, \ell_t y_t \leq x_t \leq u_t y_t, t \in N\}.$$

These sets are very common structures that occur after the aggregation of variables and/or constraints of more complex fixed charge capacitated network flow sets.

The single node flow sets have been studied for more than three decades. Padberg et al. [9] studied the case where the y_i are binary and the ℓ_j are null. They introduced the so called *flow cover inequalities* and showed this class of valid inequalities suffices to describe the convex

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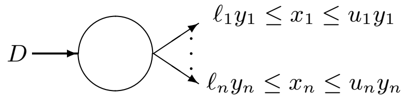

Figure 1: Single node flow problem.

hull of the feasible set when $u_j = U, \forall j \in N$. The binary case was also studied in [5]. Van Roy and Wolsey [11] derived the so called *generalized flow cover inequalities* and Stallaert [10] introduced a new class of valid inequalities by complementing binary variables. For a survey on valid inequalities for this and other related sets from the perspective of lifting see [7]. Special cases where considered by several authors. Constantino [4] describes the convex hull of several related regions, in particular, the integer single node flow set with U a large positive constant, $U > D$. Agra and Constantino [2] provide a polyhedral characterization when $\ell_j = L$ and $u_j = U$ for all j. In [3] several inequalities are extended for the case where there is a set-up variable associated to the node itself.

In this paper we describe the convex hull of the integer SNF set with two possible values for the upper bounds on each arc capacity:

$$\mathcal{X} = \{(y,x) \in \mathbb{Z}_+^{|N|} \times \mathbb{R}^{|N|} : \sum_{t \in N} x_t \le D, 0 \le x_t \le a_1 y_t, t \in N_1, 0 \le x_t \le a_2 y_t, t \in N_2\},$$

where $\{N_1, N_2\}$ define a partition of N. We assume that the coefficients a_1, a_2 and D are positive integers and $D > \max\{a_1, a_2\}$. While in the classical SNF set the y variables are binary, here they are assumed to be integer. Set $\mathcal{X}$ arises as relaxation of several fixed charge capacitated network flow sets when arc capacities may assume one of the two possible values. See Section 4 for several applications.

The description of $\mathcal{P} = conv(\mathcal{X})$ by linear inequalities is obtained from the description of integer single node flow set involving only two arcs,

$$\mathcal{Z} = \{(y_1, y_2, x_1, x_2) : x_1 + x_2 \le D,\ 0 \le x_1 \le a_1 y_1,\ 0 \le x_2 \le a_2 y_2,\ y_1, y_2 \text{ integer}\},$$

given in [1]. It has similarities with the description of the convex hull of the integer single node flow set with constant lower and upper bounds [2].

In Section 2 we summarize the results concerned with the description of the SNF problem with two arcs and, in Section 3 we introduce the lifted Euclidean inequalities to generalize those results for the SNF problem with two possible values for the upper bounds. Then, in Section 4 we test the inclusion of those inequalities in a branch and cut scheme to solve three mixed integer programs: an inventory-distribution problem, a facility location problem, and a lot-sizing multi-item problem.

2. Euclidean inequalities for the integer single node flow set with two arcs

The results in this section were published in [1].

First we consider the single node flow set with one arc, $\{(y,x) \in \mathbb{Z}_+ \times \mathbb{R}_+ : x \le D, x \le ay\}$. The convex hull of this set is completely described by the inequalities $x \ge 0$, $x \le ay$, $x \le D$, and $x - \gamma y \le (a-\gamma)\lfloor D/a \rfloor$, where $\gamma = D - a\lfloor D/a \rfloor$. The last inequality is the so-called Mixed Integer Rounding inequality [8].

Next we consider the set with two arcs, $\mathcal{Z}$. It is important to notice that there are only two integer variables involved in this model and so, for this particular structure, all the information needed to describe $conv(\mathcal{Z})$ can also be obtained from the 2-integer knapsack sets that result from the elimination of the continuous variables.

All the extreme points of $conv(\mathcal{Z})$ lie in the intersection of two of the following three hyperplanes defined by $x_1 = a_1y_1$, $x_2 = a_2y_2$ and $x_1 + x_2 = D$. Thus, every extreme point of $conv(\mathcal{Z})$ has to satisfy one of the following set of conditions: (i) $x_1 = a_1y_1$, $x_2 = a_2y_2$, (ii) $x_2 = a_2y_2$, $x_1 = D - x_2$, (iii) $x_1 = a_1y_1$, $x_2 = D - x_1$,

In case (i) we have $(y_1, y_2) \in \mathcal{Y}_\le$ where $\mathcal{Y}_\le = \{(y_1,y_2) \in \mathbb{Z}_+^2 : a_1y_1 + a_2y_2 \le D\}$. In case (ii), noticing that $0 \le x_1 \le a_1y_1$ imply $0 \le D - a_2y_2 \le a_1y_1$, we have $(y_1,y_2) \in \mathcal{Y}_1$ where

$$\mathcal{Y}_1 = \{(y_1,y_2) \in \mathbb{Z}_+^2 : a_1y_1 + a_2y_2 \ge D, y_2 \le D/a_2\}.$$

Note that constraint $y_2 \le D/a_2$ is implied by the non-negativity constraint $x_1 \ge 0$. Similarly, in case (iii) we have $(y_1,y_2) \in \mathcal{Y}_2$ where

$$\mathcal{Y}_2 = \{(y_1,y_2) \in \mathbb{Z}_+^2 : a_1y_1 + a_2y_2 \ge D, y_1 \le D/a_1\}.$$

Let us define $\mathcal{Y}_{1>} = \mathcal{Y}_1 \setminus \mathcal{Y}_=$ and $\mathcal{Y}_{2>} = \mathcal{Y}_2 \setminus \mathcal{Y}_=$ where $\mathcal{Y}_= = \{(y_1,y_2) \in \mathbb{Z}_+^2 : a_1y_1 + a_2y_2 = D\}$.

In [1] it is shown that all the coefficients involved in the computation of the extreme points and facets of the two dimensional polyhedra $conv(\mathcal{Y}_\le)$, $conv(\mathcal{Y}_1)$, $conv(\mathcal{Y}_2)$ can be obtained in $\mathcal{O}(log(D/\min\{a_1,a_2\}))$ elementary operations using a version of the Hirschberg and Wong's algorithm, [6]. This algorithm is based on the Euclidean Algorithm. Hence, the inequalities we describe next, and are based on these two dimensional polyhedra, are referred to as Euclidean inequalities.

First we consider the valid inequalities obtained from the lifting of facet-defining inequalities for $conv(\mathcal{Y}_\le)$ (corresponding to case (i)).

Proposition 2.1. *If $\alpha_1y_1 + \alpha_2y_2 \le \alpha$ is a valid facet-defining for $conv(\mathcal{Y}_\le)$ then the inequality*

$$\beta_1(x_1 - a_1y_1) + \beta_2(x_2 - a_2y_2) + \alpha_1y_1 + \alpha_2y_2 \le \alpha \tag{2.1}$$

is a valid facet-defining inequality for $conv(\mathcal{Z})$, where

$$\beta_1 = \max\left\{\frac{\alpha_1y_1 + \alpha_2y_2 - \alpha}{a_1y_1 + a_2y_2 - D} : (y_1,y_2) \in \mathcal{Y}_{1>}\right\} \text{ and } \beta_2 = \max\left\{\frac{\alpha_1y_1 + \alpha_2y_2 - \alpha}{a_1y_1 + a_2y_2 - D} : (y_1,y_2) \in \mathcal{Y}_{2>}\right\}.$$

Next, from the lifting of the facet defining inequalities of $conv(\mathcal{Y}_1)$ the following family of valid inequalities for $conv(\mathcal{Z})$ is obtained.

Proposition 2.2. *If $\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha$ is a valid facet-defining inequality for conv$(\mathcal{Y}_1)$ containing only points in $\mathcal{Y}_{1>}$ then the inequality*

$$\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha + \beta_1(x_1 + x_2 - D) + \beta_2(x_2 - a_2 y_2) \tag{2.2}$$

is a valid facet-defining inequality for conv$(\mathcal{Z})$, where

$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in \mathcal{Y}_<\right\} \text{ and } \beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in \mathcal{Y}_{2>}\right\}.$$

Finally we consider the lifting of the facet defining inequalities of *conv*$(\mathcal{Y}_2)$.

Proposition 2.3. *If $\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha$ is a valid facet-defining inequality for conv$(\mathcal{Y}_2)$ containing only points in $\mathcal{Y}_{2>}$ the inequality*

$$\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha + \beta_1(x_1 - a_1 y_1) + \beta_2(x_1 + x_2 - D) \tag{2.3}$$

is a valid facet-defining inequality for conv$(\mathcal{Z})$, where

$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in \mathcal{Y}_{1>}\right\} \text{ and } \beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in \mathcal{Y}_<\right\}.$$

In [1] it is shown that the lifting coefficients β_1 and β_2 in each Euclidean inequality (2.1), (2.2), (2.3), can be obtained directly (in constant time) from the information required to derive the corresponding two-dimensional polyhedra *conv*$(\mathcal{Y}_\leq)$, *conv*$(\mathcal{Y}_1)$, *conv*$(\mathcal{Y}_2)$. So all the coefficients involved in the Euclidean inequalities can be obtained in $\mathcal{O}(log(D/\min\{a_1, a_2\}))$ elementary operations.

Now we consider two unbounded facet-defining inequalities that can be obtained by the MIR procedure.

Proposition 2.4. *The inequality*

$$x_t - \gamma_t y_t \leq (a_t - \gamma_t)\lfloor D/a_t \rfloor \tag{2.4}$$

where $\gamma_t = D - a_t\lfloor D/a_t \rfloor$, and $t \in \{1, 2\}$, is valid for $\mathcal{Z}$.

Theorem 2.5. *[1] conv$(\mathcal{Z})$ is completely described by the trivial facet-defining inequalities and the families (2.1), (2.2), (2.3) and (2.4).*

Example 2.6. *Consider the set, $\mathcal{Z} = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{Z}^2_+ : x_1 + x_2 \leq 1154,\ x_1 \leq 21 y_1,\ x_2 \leq 76 y_2\}$ and the following restrictions $\mathcal{Y}_\leq = \{y \in \mathbb{Z}^2_+ : 21 y_1 + 76 y_2 \leq 1154\}$, $\mathcal{Y}_1 = \{y \in \mathbb{Z}^2_+ : 21 y_1 + 76 y_2 \geq 1154, y_2 \leq 15\}$, $\mathcal{Y}_2 = \{y \in \mathbb{Z}^2_+ : 21 y_1 + 76 y_2 \geq 1154, y_1 \leq 54\}$. The polyhedral description of these sets was given in [1].*

conv$(\mathcal{Y}_\leq) = \{y \in \mathbb{R}^2_+ : y_1 + 3y_2 \leq 54,\ 2y_1 + 7y_2 \leq 109,\ 5y_1 + 18y_2 \leq 274, 3y_1 + 11y_2 \leq 166,\ y_1 + 4y_2 \leq 60\}$. *From Proposition 2.1 we obtain the following facet-defining Euclidean* *inequalities.*

$$
\begin{aligned}
y_1 + 3y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{14}(x_2 - 76y_2) &\leq 54 \\
2y_1 + 7y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{6}(x_2 - 76y_2) &\leq 109 \\
5y_1 + 18y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{3}(x_2 - 76y_2) &\leq 274 \\
3y_1 + 11y_2 + \frac{1}{2}(x_1 - 21y_1) + \frac{1}{2}(x_2 - 76y_2) &\leq 166 \\
y_1 + 4y_2 + \frac{1}{7}(x_1 - 21y_1) + \frac{1}{7}(x_2 - 76y_2) &\leq 60
\end{aligned}
$$

$conv(\mathcal{Y}_1) = \{y \in \mathbb{R}^2_+ : 8y_1 + 29y_2 \geq 440,\ 5y_1 + 18y_2 \geq 274,\ 2y_1 + 7y_2 \geq 107\}$. *Only* $2y_1 + 7y_2 \geq 107$ *defines a non-trivial facet that includes only points in* $\mathcal{Y}_{1>}$. *Based on that inequality we obtain:*

$$2y_1 + 7y_2 \geq 107 + \frac{1}{6}(x_1 + x_2 - D) + 0(x_2 - 76y_2).$$

$conv(\mathcal{Y}_2) = \{y \in \mathbb{R}^2_+ : y_2 \geq 1,\ y_1 + 4y_2 \geq 56,\ 3y_1 + 11y_2 \geq 166,\ 5y_1 + 18y_2 \geq 274,\ 2y_1 + 7y_2 \geq 107,\ y_1 + y_2 \geq 16\}$. *Based on these inequalities we derive the following set of facet-defining Euclidean inequalities for* $conv(\mathcal{Z})$:

$$
\begin{aligned}
y_2 &\geq 1 + \frac{1}{20}(x_1 + x_2 - 1154) + \frac{1}{1}(x_1 - 21y_1), \\
y_1 + 4y_2 &\geq 56 + \frac{1}{7}(x_1 + x_2 - 1154) + \frac{1}{1}(x_1 - 21y_1), \\
2y_1 + 7y_2 &\geq 107 + \frac{1}{6}(x_1 + x_2 - 1154) + 0(x_1 - 21y_1), \\
y_1 + y_2 &\geq 16 + \frac{1}{14}(x_1 + x_2 - 1154) + 0(x_1 - 21y_1).
\end{aligned}
$$

3. General model

In this section we give a description of $\mathcal{P} = conv(\mathcal{X})$ by linear inequalities.

3.1. Valid inequalities that can be obtained directly from the two arc model

The following main result uses the fact that we are dealing with integer (not binary) variables. It establishes the link between sets with dimension determined by the number of different coefficients with sets in higher dimensions.

Lemma 3.1. *Consider nonempty sets* $S_1, \ldots, S_k$, *with* $S_i \subset N_1$, $i \in \{1, \ldots, k\}$, *and* $S_i \cap S_j = \emptyset$, *for all* $i, j \in \{1, \ldots, k\}$, *such that* $i \neq j$. *Consider nonempty sets* $S_{k+1}, \ldots, S_p$, *with* $S_i \subset N_2$,

$i \in \{k+1,\ldots,p\}$, and $S_i \cap S_j = \emptyset$, for all $i,j \in \{k+1,\ldots,p\}$, such that $i \neq j$. Consider the set

$$\Sigma_{\{k,p\}} = \{(Y_1,\ldots,Y_p,X_1,\ldots,X_p) \in \mathbb{Z}_+^p \times \mathbb{R}^p : \sum_{t=1}^{p} X_t \leq D,$$

$$0 \leq X_t \leq a_1 Y_t, t \in \{1,\ldots,k\}, 0 \leq X_t \leq a_2 Y_t, t \in \{k+1,\ldots,p\}\}.$$

(i) The inequality

$$\sum_{t=1}^{p} \alpha_t \sum_{j \in S_t} y_j + \sum_{t=1}^{p} \beta_t \sum_{j \in S_t} x_j \leq \alpha \tag{3.1}$$

is valid for $\mathcal{X}$ iff

$$\sum_{t=1}^{p} \alpha_t Y_t + \sum_{t=1}^{p} \beta_t X_t \leq \alpha \tag{3.2}$$

is valid for $\Sigma_{\{k,p\}}$.
(ii) If (3.1) and (3.2) are valid for $\mathcal{X}$ and $\Sigma_{\{k,p\}}$, respectively, then (3.1) defines a facet of $\mathcal{P}$ iff (3.2) defines a facet of $conv(\Sigma_{\{k,p\}})$.

Proof: Since (i) can be easily checked, we only show (ii). We assume that (3.1) and (3.2) are valid for $\mathcal{X}$ and $\Sigma_{\{k,p\}}$, respectively. Suppose (3.1) defines a facet, $\mathcal{F}$, of $\mathcal{P}$ and (3.2) does not define a facet of $conv(\Sigma_{\{k,p\}})$. Let $\mathcal{F}^a$ be the face of $conv(\Sigma_{\{k,p\}})$ defined by (3.2). Since $\mathcal{F}^a$ is not a facet of $conv(\Sigma_{\{k,p\}})$ there must exist a valid inequality, which is not a multiple of (3.2) (observe that both polyhedra are full dimensional), for $conv(\Sigma_{\{k,p\}})$, $\sum_{t=1}^{p} \alpha'_t Y_t + \sum_{t=1}^{p} \beta'_t X_t \leq \alpha'$ satisfied as equation by every point in $\mathcal{F}^a$. Hence $\sum_{t=1}^{p} \alpha'_t \sum_{j \in S_t} y_j + \sum_{t=1}^{p} \beta'_t \sum_{j \in S_t} x_j \leq \alpha'$ is not a multiple of (3.1), and, by (i), is valid for $\mathcal{P}$. In order to obtain a contradiction, it suffices to prove that each point in $\mathcal{F}$ also satisfies this inequality as equation. Suppose not, that is, there is $(y',x') \in \mathcal{F}$ satisfying $\sum_{t=1}^{p} \alpha'_t \sum_{j \in S_t} y_j + \sum_{t=1}^{p} \beta'_t \sum_{j \in S_t} x_j < \alpha'$. Then a contradiction is obtained by considering the point $(Y',X') = (\sum_{j \in S_1} y'_j, \ldots, \sum_{j \in S_p} y'_j, \sum_{j \in S_1} x'_j, \ldots, \sum_{j \in S_p} x'_j)$ that belongs to $\mathcal{F}^a$ and verifies $\sum_{t=1}^{p} \alpha'_t Y'_t + \sum_{t=1}^{p} \beta'_t X'_t < \alpha'$.

Now, suppose (3.2) defines a facet of $conv(\Sigma_{\{k,p\}})$ and let $\mathcal{F}$ be the face defined by (3.1). Consider a generic facet for $\mathcal{P}$ defined by

$$\sum_{j \in N} \mu_j y_j + \sum_{j \in N} \nu_j x_j \leq \pi \tag{3.3}$$

containing $\mathcal{F}$. We will show that (3.3) is a multiple of (3.1).

Since both polyhedra are full dimensional, the facet of $conv(\Sigma_{\{k,p\}})$ defined by (3.2) must contain $2p$ affinely independent points, $(Y^j, X^j), j = 1,\ldots,2p$. Construct $2p$ points in $\mathcal{F}$, $(\overline{y}^j, \overline{x}^j)$ as follows: For each $i \in \{1,\ldots,p\}$, set $\overline{y}_t^j = Y_i^j$ and $\overline{x}_t^j = X_i^j$ for $t = t_i = argmin\{j \in S_i\}$ and $\overline{y}_t^j = \overline{x}_t^j = 0$, otherwise.

Consider the point $(\overline{y}^1, \overline{x}^1)$, and for each $t \in N \setminus S$, with $S = \bigcup_{i=1}^{p} S_i$, consider the point (y^t, x^t) belonging to $\mathcal{F}$, where (y^t, x^t) coincides with $(\overline{y}^1, \overline{x}^1)$ for all coordinates except for

$y_i^t = 1$ when $i = t$. As $(\overline{y}^1, \overline{x}^1)$ and (y^t, x^t), satisfy (3.3) as equation, it follows that $\mu_t = 0$. Hence we have $\mu_t = 0, \forall t \in N \setminus S$.

We assume $(\overline{y}^1, \overline{x}^1)$ satisfies $\sum_{k\in N} \overline{x}_k^1 < D$. Next, consider the point $(\overline{y}^1, \overline{x}^1)$ and for each $t \in N \setminus S$, consider (y'^t, x'^t) belonging to $\mathcal{F}$ where (y'^t, x'^t) coincides with $(\overline{y}^1, \overline{x}^1)$ for all coordinates except for $y_i'^t = 1, x_i'^t = \epsilon$ when $i = t$, where $t \in N \setminus S$, and $0 < \epsilon < \min\{a_1, a_2, D - \sum_{j\in N} \overline{x}_j^1\}$. As $(\overline{y}^1, \overline{x}^1)$ and (y'^t, x'^t), satisfy (3.3) as equation, it follows that $\nu_t = 0, \forall t \in N \setminus S$.

For each $t \in \{1, \ldots, p\}$, consider two affinely independent points $(Y^a, X^a), (Y^b, X^b)$ with $Y_t^a > 0, Y_t^b > 0$ and assume that $(Y_t^a, X_t^a), (Y_t^b, X_t^b)$ are affinely independent (observe that such two points must exist since (3.2) defines a facet). As before, define $(\overline{y}^a, \overline{x}^a)$ and $(\overline{y}^b, \overline{x}^b)$ as follows: for each $i \in \{1, \ldots, p\}$, set $\overline{y}_t^a = Y_i^a$ and $\overline{x}_t^a = X_i^a$ for $t = t_i$ and $\overline{y}_t^j = \overline{x}_t^j = 0$, otherwise. Similarly for $(\overline{y}^b, \overline{x}^b)$.

Now, for each $i \in \{1, \ldots, p\}$, and each $k \in S_i \setminus \{t_i\}$ we construct two points (y'^a, x'^a) and (y'^b, x'^b) in $\mathcal{F}$, such that $y_t'^a = \overline{y}_t^a$, $x_t'^a = \overline{x}_t^a$, for $t \neq t_i$ and $t \neq k$, and $y_{t_i}'^a = x_{t_i}'^a = 0$ and $y_k'^a = \overline{y}_{t_i}^a$, $x_k'^a = \overline{x}_{t_i}^a$. Point (y'^b, x'^b) is constructed similarly. Using these points we obtain $\mu_i = \overline{\alpha}_t$, $\nu_i = \overline{\beta}_t, \forall i \in S_t, t \in \{1, 2\}$.

Finally, since the $2p$ points in $\mathcal{F}$, $(\overline{y}^j, \overline{x}^j)$, are affinely independent (as all its non-null components coincide with the non-null components of the affinely independent points (Y^j, X^j)), then we conclude that (3.3) must be a multiple of (3.1). □

Lemma 3.1 states that from each facet defining inequality of $conv(\mathcal{Z})$ we obtain facet defining inequalities for $\mathcal{P}$.

Corollary 3.2. *The inequality*

$$\alpha_1 y_1 + \alpha_2 y_2 + \beta_1 x_1 + \beta_2 x_2 \leq \alpha \tag{3.4}$$

defines a facet of $conv(\mathcal{Z})$ if and only if

$$\alpha_1 \sum_{j\in S_1} y_j + \alpha_2 \sum_{j\in S_2} y_j + \beta_1 \sum_{j\in S_1} x_j + \beta_2 \sum_{j\in S_2} x_j \leq \alpha. \tag{3.5}$$

defines a facet for $\mathcal{P}$ for all $\emptyset \neq S_1 \subseteq N_1, \emptyset \neq S_2 \subseteq N_2$.

3.2. Lifted Euclidean Inequalities

In this section we present the families of valid inequalities necessary for the description of $\mathcal{P}$ that cannot be obtained directly from the aggregated model $\mathcal{Z}$.

The following family of valid inequalities is necessary when the equation $a_1 Y_1 + a_2 Y_2 = D$ has no nonnegative integer solutions.

Proposition 3.3. *Let* $\gamma = D - \max\{a_1Y_1 + a_2Y_2 : (Y_1, Y_2) \in \mathcal{Y}_{\leq}\}$, $\epsilon_1 = min\{a_1Y_1 + a_2Y_2 : (Y_1, Y_2) \in \mathcal{Y}_1\} - D$, *and* $\epsilon_2 = min\{a_1Y_1 + a_2Y_2 : (Y_1, Y_2) \in \mathcal{Y}_2\} - D$. *If* $\gamma > 0$, *then the following inequality is valid for* $\mathcal{X}$,

$$a_1 \sum_{j\in S_1} y_j + a_2 \sum_{j\in S_2} y_j + \frac{\gamma + \epsilon_1}{\epsilon_1} \sum_{j\in S_1}(x_j - a_1y_j) + \frac{\gamma + \epsilon_2}{\epsilon_2} \sum_{j\in S_2}(x_j - a_2y_j) + \sum_{j\in N\setminus(S_1\cup S_2)} (x_j - \gamma y_j) \leq D - \gamma \tag{3.6}$$

where $S_1 \subset N_1$ *and* $S_2 \subset N_2$.

Observe that $\gamma > 0$ implies $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Conversely, the two conditions $\epsilon_1 > 0, \epsilon_2 > 0$ imply $\gamma > 0$. The proof of Proposition 3.3 is left to the Appendix.

Next we explain how to obtain facet-defining inequalities from lifting of the Euclidean inequalities. For ease of notation, for a subset S we will denote by $X(S) = \sum_{j\in S} x_j$ and $Y(S) = \sum_{j\in S} y_j$.

Let Λ denote the set of the coefficients of all Euclidean facet-defining inequalities of types (2.1), (2.2), (2.3): $\Lambda = \{(\mu_1, \mu_2, \nu_1, \nu_2, \lambda) : \mu_1y_1 + \mu_2y_2 + \nu_1x_1 + \nu_2x_2 \leq \lambda$ is a facet-defining inequality of one of the types (2.1), (2.2), (2.3) for $Z\}$.

Consider an inequality for the restriction of $\mathcal{X}$ to $S_1 \cup S_2, S_1 \subseteq N_1, S_2 \subseteq N_2$, obtained from the Euclidean inequality defined by $(\mu_1, \mu_2, \nu_1, \nu_2, \lambda) \in \Lambda$:

$$\mu_1Y(S_1) + \mu_2Y(S_2) + \nu_1X(S_1) + \nu_2X(S_2) \leq \lambda. \tag{3.7}$$

From Lemma 3.1 it follows that (3.7) defines a facet of the restricted set resulting from X by considering only the subset of variables in $S_1 \subseteq N_1$ and $S_2 \subseteq N_2$.

We assume henceforward $\frac{\mu_2 + a_2\nu_2}{\mu_1 + a_1\nu_1} \neq \frac{a_2}{a_1}$. It can be shown that the lifting of inequality (3.7) when $\frac{\mu_2 + a_2\nu_2}{\mu_1 + a_1\nu_1} = \frac{a_2}{a_1}$ gives inequalities (3.6). We omit the proof of this statement since it is too technical and this result will not be used in the remaining of the paper.

Next we discuss the computation of the lifting coefficients of the variable pairs (y_j, x_j), for $j \in I = I_1 \cup I_2$, with $I_1 \in N_1 \setminus S_1$, $I_2 \in N_2 \setminus S_2$, when $\frac{\mu_2 + a_2\nu_2}{\mu_1 + a_1\nu_1} \neq \frac{a_2}{a_1}$ in order to obtain a valid inequality for $\mathcal{X}$:

$$\mu_1 \sum_{j\in S_1} y_j + \mu_2 \sum_{j\in S_2} y_j + \sum_{j\in I} \theta_j y_j + \nu_1 \sum_{j\in S_1} x_j + \nu_2 \sum_{j\in S_2} x_j + \sum_{j\in I} \xi_j x_j \leq \lambda \tag{3.8}$$

The lifting function associated with inequality (3.7) is given by

$$\begin{aligned} \phi(z) \quad = \quad & \min \lambda - \mu_1Y(S_1) - \mu_2Y(S_2) - \nu_1X(S_1) - \nu_2X(S_2) && (3.9)\\ \text{s. to} \quad & \sum_{j\in S_1\cup S_2} x_t \leq D - z && (3.10)\\ & 0 \leq x_j \leq a_ty_j, t \in \{1, 2\}, j \in S_t && (3.11)\\ & y_j \in \mathbb{Z}_+, j \in S_1 \cup S_2 && (3.12) \end{aligned}$$

The lifing function of the following Euclidean inequality given in Example 2.6,

$$2y_1 + 7y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{6}(x_2 - 76y_2) \le 109 \Leftrightarrow -114y_1 - 34y_2 + 6x_1 + x_2 \le 654$$

is depicted in Figure 2.

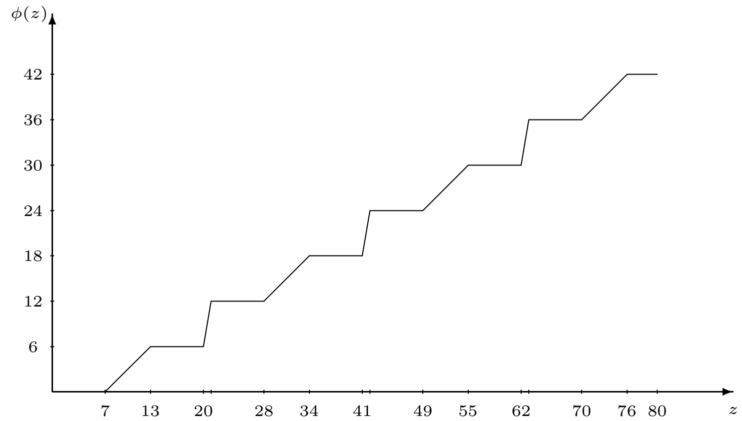

Figure 2: The lifting function ϕ associated with $-114y_1 - 34y_2 + 6x_1 + x_2 \le 654$, for $z \in [0, 80]$.

The set of possible lifting coefficients for inequality (3.8) is denoted by $\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}$ and can be rewritten as

$$\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)} = \{(\theta, \xi) \in \mathbb{R}^{2|I|} : \sum_{k\in I} \theta_k y_k + \sum_{k\in I} \xi_k x_k \le \phi(\sum_{k\in I} x_k), \forall (y, x) \in \mathcal{X}(I)\}$$

where

$$\mathcal{X}(I) = \{(x, y) \in: \mathbb{R}^{|I|}_+ \times \mathbb{Z}^{|I|}_+ : X(I) \le D, x_j \le a_1 y_j, j \in I_1, x_j \le a_2 y_j, j \in I_2\}.$$

The interesting cases are those where $(\theta_j, \xi_j) \neq (0, 0), \forall j \in I$, $(\theta_j, \xi_j) \neq (\mu_1, \nu_1)$ if $j \in I_1$ and $(\theta_j, \xi_j) \neq (\mu_2, \nu_2)$ if $j \in I_2$.

In order to compute the lifting coefficients only extreme points and extreme rays of $\mathcal{P}$ need to be considered in the definition of $\mathcal{X}(I)$. Extreme rays are of the form $(0, e_j)$ where e_j is the unit vector with one in position j and zero elsewhere. Extreme rays imply $\theta_j \le 0, j \in I$. The following lemma considers extreme points of $\mathcal{P}$.

Lemma 3.4. *Assume* (3.8) *defines a facet of* $\mathcal{P}$, *with* $(\mu_1, \mu_2, \nu_1, \nu_2) \in \Lambda$, *and* $(\mu_1, \nu_1) \neq (\theta_j, \eta_j), j \in I_1$, $(\mu_2, \nu_2) \neq (\theta_j, \eta_j), j \in I_2$. *If* (x, y) *is a tight extreme point of* $\mathcal{P}$ *in the face defined by* (3.8) *and if* $y_k > 0$, *for some* $k \in I_1 \cup I_2$, *then (i)* $y_k = 1$; *(ii)* $y_j = 0, j \in I_1 \cup I_2 \setminus \{k\}$; *(iii)* $0 < x_k = D - X(S_1 \cup S_2) < a_t$ *where* $t = 1$ *if* $k \in I_1$ *and* $t = 2$ *if* $k \in I_2$; *(iv)* $X(S_1) = a_1 Y(S_1)$; $X(S_2) = a_2 Y(S_2)$.

The proof is given in the Appendix.

We can restrict $\mathcal{X}(I)$ to the set of points obtained by projecting the extreme points of $\mathcal{P}$ into the space of the lifting variables. We denote the restricted set as $\overline{\mathcal{X}}(I)$. Using (i) and (ii) from Lemma 3.4 we can write set $\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}$ as follows.

$$\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)} = \{(\theta,\xi) \in \mathbb{R}_-^{|I|} \times \mathbb{R}^{|I|} : \theta_k + \xi_k x_k \leq \phi(x_k), \forall (y,x) \in \overline{\mathcal{X}}(I), y_k = 1, k \in I\}.$$

From condition (iii) and (iv), if (x,y) is an extreme point of $\mathcal{P}$ and $y_k = 1, k \in I$, then $0 \leq x_k = D - X(S_1 \cup S_2) = D - a_1Y(S_1) - a_2Y(S_2) \leq a_t$. Hence $D - a_t \leq a_1Y(S_1) + a_2Y(S_2) \leq D$.

For $t \in \{1,2\}$, let us define $\Sigma^t = \{(Y_1,Y_2) \in \mathbb{Z}_+^2 : D - a_t \leq a_1Y_1 + a_2Y_2 \leq D\}$; $V(\Sigma^t)$ as the set of extreme points of $conv(\Sigma^t)$, and $\Gamma^t = \{\gamma : \gamma = D - a_1Y_1 - a_2Y_2$ for some $(Y_1,Y_2) \in V(\Sigma^t)\}$. Then

Proposition 3.5.

$$\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)} = \{(\theta,\xi) \in \mathbb{R}_-^{|I|} \times \mathbb{R}^{|I|} : \theta_j + \xi_j\gamma \leq \phi(\gamma), \forall t \in \{1,2\}, j \in I_t, \gamma \in \Gamma^t\}.$$

Since each inequality defining $\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}$ involves only one variable pair, set $\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}$ can be decomposed accordingly to variables pairs:

$$\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)} = \times_{j\in I}\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}(j)$$

where

$$\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}(j) = \{(\theta_j,\xi_j) \in \mathbb{R}_- \times \mathbb{R} : \theta_j + \xi_j\gamma \leq \phi(\gamma), t \in \{1,2\}, j \in I_t, \gamma \in \Gamma^t\}.$$

Henceforward we focus on computing the lifting coefficients from $\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}(j)$ only. We can observe that for each $t \in \{1,2\}$, the set $\Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}(j)$ is always the same for every $j \in I_t$. Hence, for $t \in \{1,2\}$, we define $\Theta^t_{(\mu_1,\mu_2,\nu_1,\nu_2)} = \{(\theta,\xi) \in \mathbb{R}_- \times \mathbb{R} : \theta + \xi\gamma \leq \phi(\gamma), \gamma \in \Gamma^t\} = \Theta_{(\mu_1,\mu_2,\nu_1,\nu_2)}(j), j \in I_t$.

It is well known that only extreme points of $\Theta^t_{(\mu_1,\mu_2,\nu_1,\nu_2)}$ lead to facet defining inequalities of type (3.8). Since $\Theta^t_{(\mu_1,\mu_2,\nu_1,\nu_2)}, t \in \{1,2\}$ are two dimensional polyhedron the set of extreme points can be computed efficiently [6].

The set of extreme points of $\Theta^t_{(\mu_1,\mu_2,\nu_1,\nu_2,\lambda)}$ will be denoted by $\overline{\Theta}^t_{(\mu_1,\mu_2,\nu_1,\nu_2,\lambda)} = \{(\theta_t^k,\xi_t^k : k \in T_t\}$ where $T_t = \{1,\ldots,n_t\}$ and n_t denotes the number of extreme points which is $\mathcal{O}(log(D/\min\{a_1,a_2\}))$. These sets can be computed in the same time complexity.

Example 3.6. *Consider the mixed integer set* $\mathcal{X} = \{(y,x) \in \mathbb{Z}_+^3 \times \mathbb{R}_+^3 : x_1 + x_2 + x_3 \leq 1154, x_1 \leq 21y_j, x_2 \leq 76y_2, x_3 \leq 76y_3\}$. *The full polyhedral description for the restricted set with* $x_3 = y_3 = 0$ *was given in Example 2.6. Next we discuss the lifting of variable pair* (x_3,y_3). *Hence* $I_1 = \emptyset$ *and* $I_2 = \{3\}$.

Observe that Σ^1 *and* Σ^2 *do not depend on the particular facet-defining inequality we are considering. Figure 3 depicts the set*

$$V(\Sigma^2) = \{(54,0),(51,1),(44,3),(26,8),(4,14),(0,15),(8,12),(26,7),(48,1),(52,0)\}.$$

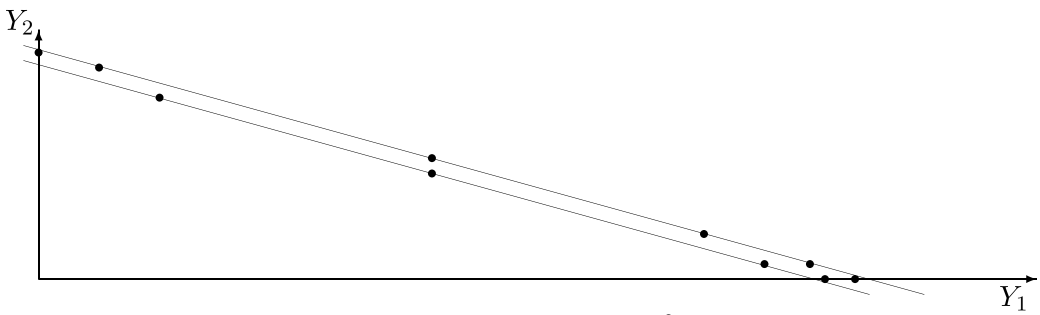

Figure 3: Extreme points of $conv(\Sigma^2)$ derived from set X.

Now consider the Euclidean inequality

$$2y_1 + 7y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{6}(x_2 - 76y_2) \leq 109$$

which is equivalent to

$$-114y_1 - 34y_2 + 6x_1 + x_2 \leq 654$$

Based on the points in $V(\Sigma^2)$ *we obtain* $\Gamma^t = \{20, 7, 2, 0, 6, 14, 74, 76, 70, 62\}$ *and (see Figure 2)* $\phi(20) = 6$, $\phi(7) = 0$, $\phi(2) = 0$, $\phi(0) = 0$, $\phi(6) = 0$, $\phi(14) = 6$, $\phi(74) = 40$, $\phi(76) = 42$, $\phi(70) = 36$, $\phi(62) = 30$. *Thus* $\Theta^2_{(\mu_1,\mu_2,\nu_1,\nu_2,\lambda)} = \{(\theta, \xi) \in \mathbb{R}_- \times \mathbb{R} : \theta + 20\xi \leq 6, \theta + 7\xi \leq 0, \theta+2\xi \leq 0, \theta \leq 0, \theta+6\xi \leq 0, \theta+14\xi \leq 6, \theta+74\xi \leq 40, \theta+76\xi \leq 42, \theta+70\xi \leq 36, \theta+62\xi \leq 30\}$. *The extreme points are:*

$$\overline{\Theta}^2_{(\mu_1,\mu_2,\nu_1,\nu_2,\lambda)} = \{(-34, 1), (-\frac{42}{13}, \frac{6}{13}), (-\frac{38}{7}, \frac{4}{7}), (-\frac{33}{2}, \frac{3}{4}), (0, 0)\}.$$

Hence we obtain the following set of facet-defining inequalities with $(\theta, \xi) \neq (0, 0)$ *and* $(\theta, \xi) \neq (-34, 1)$*:*

$$\begin{aligned}
-114y_1 - 34y_2 - \frac{42}{13}y_3 + 6x_1 + x_2 + \frac{6}{13}x_3 &\leq 654 \\
-114y_1 - 34y_2 - \frac{38}{7}y_3 + 6x_1 + x_2 + \frac{4}{7}x_3 &\leq 654 \\
-114y_1 - 34y_2 - \frac{33}{2}y_3 + 6x_1 + x_2 + \frac{3}{4}x_3 &\leq 654
\end{aligned}$$

Now we summarize the above discussion.

Proposition 3.7. *Consider* $(\mu_1, \mu_2, \nu_1, \nu_2, \lambda) \in \Lambda$ *and consider the disjoint sets* S_1, I_1, $\emptyset \neq S_1 \subset N_1$, $\emptyset \neq I_1 \subset N_1$, *and* S_2, I_2, $\emptyset \neq S_2 \subset N_2$, $\emptyset \neq I_2 \subset N_2$. *For* $t \in \{1, 2\}$, *let* $\{I_t^1, \ldots, I_t^{n_t}\}$ *define a partition of* I_t. *Then the following lifted Euclidean inequality is valid for* X *:*

$$\mu_1 \sum_{j\in S_1} y_j + \mu_2 \sum_{j\in S_2} y_j + \sum_{k\in T_1} \sum_{j\in I_1^k} \theta_1^k y_j + \sum_{k\in T_2} \sum_{j\in I_2^k} \theta_2^k y_j + \nu_1 \sum_{j\in S_1} x_j + \nu_2 \sum_{j\in S_2} x_j + \sum_{k\in T_1} \sum_{j\in I_1^k} \xi_1^k x_j + \sum_{k\in T_2} \sum_{j\in I_2^k} \xi_2^k x_j \leq \lambda \tag{3.13}$$

where $(\theta_t^k, \xi_t^k) \in \Theta^t_{(\mu_1,\mu_2,\nu_1,\nu_2,\lambda)}$ *for* $t \in \{1, 2\}, k \in T_t$.

Note that all the coefficients involved in these valid inequalities can be computed in polynomial time since the information required can be obtained by computing the extreme points of two dimensional polyhedra, using a version of the Hirschberg and Wong's algorithm (which is based on the Euclidean Algorithm).

3.3. The convex hull

Here we establish the main result of this paper.

Theorem 3.1. *The inequalities* $x_j \geq 0, j \in N$, $x_j \leq a_1 y_j, j \in S$, $x_j \leq a_2 y_j, j \in N \setminus S$, $\sum_{j \in N} x_j \leq D$, (3.6) *and the lifted Euclidean inequalities (3.13) suffice to describe* $P = conv(X)$.

Proof: To prove this theorem we assume that the inequality

$$\sum_{j \in N} \mu_j y_j + \sum_{j \in N} \nu_j x_j \leq \lambda \tag{3.14}$$

defines a non-trivial facet, $\mathcal{F}_{(\mu,\nu,\lambda)}$, of P (we call inequalities $x_j \geq 0, j \in N$, $x_j \leq a_1 y_j, j \in S$, $x_j \leq a_2 y_j, j \in N \setminus S$, $\sum_{j \in N} x_j \leq D$, trivial facets) and show that either it belongs to one of the families for the two dimensional case (3.5) or it belongs to one of the families (3.6) or (3.13) (which includes inequalities (3.5)). From Lemma 3.1 we may assume w.l.o.g. $(\mu_j, \nu_j) \neq (0,0), j \in N$.

We will use the following notation for each $t \in \{1,2\}$ such that $N_t \neq \emptyset$, $\ell_t \in arg \min\{\mu_j, j \in N_t\}$, $S_t = \{j \in N_t : \mu_j = \mu_{\ell_t}\}$, $I_t = N_t \setminus S_t$ and $I = I_1 \cup I_2$. We also define the following aggregated variables: $Y_t = Y(S_t), X_t = X(S_t), t \in \{1,2\}$.

As the null vector is in X then $\lambda > 0$.

The following properties hold:

(i) For all $j \in N_1 \cup N_2, \mu_j < 0$.

(ii) For all $j \in N_1 \cup N_2, \nu_j > 0$.

(iii) (a) $\nu_{\ell_1} > \nu_j$ for all $j \in N_1 \setminus S_1$; (b) $\nu_{\ell_2} > \nu_j$ for all $j \in N_2 \setminus S_2$.

(iv) (a) $\mu_{\ell_1} + a_1 \nu_{\ell_1} > \mu_j + a_1 \nu_j$ for all $j \in N_1 \setminus S_1$; (b) $\mu_{\ell_2} + a_2 \nu_{\ell_2} > \mu_j + a_2 \nu_j$ for all $j \in N_2 \setminus S_2$.

(v) If $(x^*, y^*) \in \mathcal{F}_{(\mu,\nu,\lambda)} \cap X$ then $\sum_{j \in I} y_j^* \leq 1$.

As the unit vector with 1 in position corresponding to y_j and 0 elsewhere is a ray of P, then $\mu_j \leq 0$. If $\mu_j = 0$ then as we are assuming $(\mu_j, \nu_j) \neq (0,0)$ we must have $\nu_j > 0$. Then each point in the facet $\mathcal{F}$ must satisfy $\sum_{j \in N} x_j = D$, (since otherwise, if there is a point $(x,y) \in \mathcal{F}$ with $\sum_{j \in N} x_j < D$, then we can increase the value of y_j and x_j obtaining a violated point) which is a contradiction. Thus condition (i) must hold.

Property (ii) follows from the fact that if $\nu_j \leq 0$, then $x_j = y_j = 0$ for every point in $\mathcal{F}$ (otherwise, if there is a solution $(x,y) \in \mathcal{F}$ with $y_j > 0$ a new point in X can be constructed from (x,y) by decreasing the value of x_j and y_j to zero which violates (3.14)).

To prove (iii) (a), observe that if $\nu_{\ell_1} \leq \nu_j$ for some $j \in N_1 \setminus S_1$, then $y_j = x_j = 0$ for every point in $\mathcal{F}$. Similarly for (b).

To prove (iv), suppose $\mu_{\ell_1} + a_1\nu_{\ell_1} \leq \mu_j + a_1\nu_j$ for some $j \in N_1 \setminus S_1$. Then $\mu_{\ell_1} + \gamma\nu_{\ell_1} < \mu_j + \gamma\nu_j$ for each $0 < \gamma < a_1$. This implies that for each $(x, y) \in \mathcal{F}$ either $x_{\ell_1} = 0$ if $y_{\ell_1} = 0$ or $x_{\ell_1} = a_1 y_{\ell_1}$ if $y_{\ell_1} > 0$ (otherwise, if there is a solution $(x, y) \in \mathcal{F}$ with $x_{\ell_1} < a_1 y_{\ell_1}$ then a violated point can be obtained by transferring the values of x_{ℓ_1} and y_{ℓ_1} to variables x_j and y_j, respectively). In each case the condition $x_{\ell_1} = a_1 y_{\ell_1}$ holds, which is a contradiction.

To prove (v) suppose $y^*_j \geq 2$ with $j \in I_1$. If $x^*_j \leq a_1(y^*_j - 1)$ a new point violating (3.14) can be obtained by decreasing y^*_j since, by (i), $\mu_j < 0$. If $x^*_j > a_1(y^*_j - 1)$ a new point violating (3.14) can be obtaining, using (iv), by decreasing x^*_j, y^*_j and increasing $x^*_{\ell_1}, y^*_{\ell_1}$ in the same amounts. The case with $j \in I_2$ is similar. Finally the case where $\sum_{j \in I} y^*_j > 1$ and $y^*_j \leq 1, \forall j \in I$ can be reduced to one of the previous ones by changing, appropriately, the values of the x variables.

First notice that if $I = \emptyset$ then, from Lemma 3.1, (3.14) is one of the valid inequalities (3.5) obtained from the aggregated model with two integer variables which is a particular case of (3.13).

Next we show that if $N_1 = \emptyset$ or $N_2 = \emptyset$, then inequality (3.14) is an inequality (3.1) obtained from one of the inequalities (2.4).

Suppose $N_1 = \emptyset$. Next we show that μ_j, and ν_j, are constant for all $j \in N_2$. Let $j^* \in argmax\{\mu_j + a_2\nu_j : j \in N_2\}$. There must exist a point (x, y) in $\mathcal{F}_{(\mu,\nu,\lambda)}$ satisfying $\sum_{j \in N} x_j < D$. This point must satisfy $x_j = a_2 y_j$, $\forall j \in N_2$ since $\nu_j > 0, j \in N_2$. Thus, $\lambda = \lfloor D/a_2 \rfloor(\mu_{j^*} + a_2\nu_{j^*})$. This implies that $\mu_j + \gamma_2\nu_j \leq 0$ for all $j \in N_2$, where $\gamma_2 = D - a_2\lfloor D/a_2 \rfloor$, since otherwise, if $\mu_j + \gamma_2\nu_j > 0$, then the point obtained by setting $y_{j^*} = \lfloor D/a_2 \rfloor$, $x_{j^*} = a_2 y_{j^*}$, $y_j = 1$, $x_j = \gamma_2$, and the remaining variables to zero, violates (3.14). Hence, if there is $j \in N_2$ such that $\mu_j + a_2\nu_j < \mu_{j^*} + a_2\nu_{j^*}$, then each point in $\mathcal{F}_{(\mu,\nu,\lambda)}$ satisfies $x_j = \gamma_2 y_j$, which is a contradiction since (3.14) defines a facet. Thus $\mu_j + a_2\nu_j = \mu_{j^*} + a_2\nu_{j^*}$ for all $j \in N_2$. It remains to prove that $\mu_{j^*} + a_2\nu_{j^*} = \mu_j + a_2\nu_j$ implies $\mu_{j^*} = \mu_j$ and $\nu_{j^*} = \nu_j$. Suppose not. Assume $\mu_{j^*} + a_2\nu_{j^*} = \mu_j + a_2\nu_j$ and $\mu_j > \mu_{j^*}$. This implies $\nu_j < \nu_{j^*}$. Thus, if there exists a tight point with $x_{j^*} < a_2 y_{j^*}$, one can obtain a feasible solution violating (3.14) by increasing x_j in $x_{j^*} - a_2(y_{j^*} - 1)$, y_j in one unit, and decreasing x_{j^*} and y_{j^*} by the same amounts. This implies $x_{j^*} = a_2 y_{j^*}$ for every point in $\mathcal{F}_{(\mu,\nu,\lambda)}$ which is a contradiction. The proof for the case $\mu_{j^*} + a_2\nu_{j^*} = \mu_j + a_2\nu_j$ and $\mu_j < \mu_{j^*}$ is similar and implies $x_j = a_2 y_j$ for every point in $\mathcal{F}_{(\mu,\nu,\lambda)}$. Thus $\mu_j = \mu_{j^*}$, $\nu_j = \nu_{j^*}$, for all $j \in N_2$. Using Lemma 3.1, we conclude that (3.14) must be obtained from a facet-defining inequality for the set with $n = 1$. Thus (3.14) belongs to family (3.5) obtained from a inequality (2.4) with $t = 2$. Similarly, if $N_2 = \emptyset$, then (3.5) results from (2.4) with $t = 1$.

Henceforth we assume $N_1 \neq \emptyset$, $N_2 \neq \emptyset$. If $I = \emptyset$, then $\mu_j = \mu_{\ell_t}$, and $\nu_j = \nu_{\ell_t}$, $j \in N_t, t \in \{1, 2\}$. From Lemma 3.1 we conclude that (3.14) must be obtained from a facet-defining inequality for the set $conv(Z)$ with $\mid N_1 \mid = \mid N_2 \mid = 1$. By Theorem (2.5) we conclude that coefficients $(\mu_{\ell_1}, \nu_{\ell_1}, \mu_{\ell_2}, \nu_{\ell_2}, \lambda)$ of (3.14) belong to Λ, that is, they result from (2.1), (2.2), (2.3).

Henceforth we assume $I \neq \emptyset$. We will also use the notation $X_t = X(S_t),\ Y_t = Y(S_t), t \in \{1, 2\}$.

Let $j \in I$ and consider a point $(y^*, x^*) \in \mathcal{F}_{(\mu,\nu,\lambda)} \cap X$ such that $y_j^* > 0$. From (v) we have $y_j^* = 1$ and from (i)-(iv) it follows $x_j^* = D - X_1^* - X_2^*$, $X_1^* = a_1 Y_1^*$, $X_2^* = a_2 Y_2^*$. Suppose $j \in I_1$. As $0 < x_j^* < a_1 y_j^*$ implies $0 < D - X_1^* - X_2^* < a_1 \Rightarrow 0 < D - a_1 Y_1^* - a_2 Y_2^* < a_1 \Rightarrow (Y_1^*, Y_2^*) \in \Sigma^1$. Since (y^*, x^*) belongs to $\mathcal{F}_{(\mu,\nu,\lambda)}$, it satisfies (3.14) at equality. Thus, setting $x_j^* = D - X_1^* - X_2^*$, $X_1^* = a_1 Y_1^*$, $X_2^* = a_2 Y_2^*$, and $x_{j'}^* = 0, j' \notin S_1 \cup S_2 \cup \{j\}$, then we have $Y_1^*(\mu_{\ell_1} + a_1\nu_{\ell_1} - a_1\nu_j) + Y_2^*(\mu_{\ell_2} + a_2\nu_{\ell_2} - a_2\nu_j) = \lambda - D\nu_j - \mu_j$. Therefore (Y_1^*, Y_2^*) must be optimal to

$$\text{(AP1): } \max\{(\mu_{\ell_1} + a_1\nu_{\ell_1} - a_1\nu_j)Y_1 + (\mu_{\ell_2} + a_2\nu_{\ell_2} - a_2\nu_j)Y_2 : (Y_1, Y_2) \in \Sigma^1\}$$

(otherwise, considering an optimal solution to AP1 we could construct a point violating (3.14)). Similarly, if $j \in I_2$ then (Y_1^*, Y_2^*) must be optimal to

$$\text{(AP2): } \max\{(\mu_{\ell_1} + a_1\nu_{\ell_1} - a_1\nu_j)Y_1 + (\mu_{\ell_2} + a_2\nu_{\ell_2} - a_2\nu_j)Y_2 : (Y_1, Y_2) \in \Sigma^2\}.$$

Case 1. There is a $i \in I$ such that

$$\mu_{\ell_1} + a_1\nu_{\ell_1} - a_1\nu_i = 0, \tag{3.15}$$

$$\mu_{\ell_2} + a_2\nu_{\ell_2} - a_2\nu_i = 0, \tag{3.16}$$

which implies $\dfrac{\mu_{\ell_2} + a_2\nu_{\ell_2}}{\mu_{\ell_1} + a_1\nu_{\ell_1}} = \dfrac{a_2}{a_1}$.

There must exist solutions (y, x) in $\mathcal{F}_{(\mu,\nu,\lambda)}$ satisfying $y_i = 0$ and $X_1 + X_2 < D$ (since $y_j > 0$ for some $j \in I$ implies $\sum_{j \in N} x_j = D$ because $\nu_j > 0$). Then $x_j = y_j = 0, j \in I$. In those cases $X_1 = a_1 Y_1$, $X_2 = a_2 Y_2$, because $\nu_{\ell_1}, \nu_{\ell_2} > 0$. Hence, $\mu_{\ell_1} Y_1 + \mu_{\ell_2} Y_2 + \nu_{\ell_1} X_1 + \nu_{\ell_2} Y_2 = \lambda \Leftrightarrow (\mu_{\ell_1} + a_1\nu_{\ell_1})Y_1 + (\mu_{\ell_2} + a_2\nu_{\ell_2})Y_2 = \lambda$. Using equations (3.15) and (3.16) it follows that $a_1\nu_i Y_1 + a_2\nu_i Y_2 = \lambda$. Noticing that $(Y_1, Y_2) \in \mathcal{Y}_{\leq}$ then $\lambda = \nu_i \max\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \mathcal{Y}_{\leq}\} \Rightarrow \lambda = \nu_i(D - \gamma)$, where $\gamma = D - \max\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \mathcal{Y}_{\leq}\}$.

There must exist solutions in $\mathcal{F}_{(\mu,\nu,\lambda)}$ satisfying $y_i = 0$ and $0 < X_1 < a_1 Y_1$ (otherwise $X_1 = a_1 Y_1$ for every point in $\mathcal{F}$). Again $x_j = y_j = 0, j \in I$. In this case it must occur $X_2 = a_2 Y_2$, $X_1 = D - X_2 = D - a_2 Y_2$. Thus,

$$\mu_{\ell_1} Y_1 + (D - a_2 Y_2)\nu_{\ell_1} + (\mu_{\ell_2} + a_2\nu_{\ell_2})Y_2 = \lambda$$

$$\Leftrightarrow \mu_{\ell_1} Y_1 + (\mu_{\ell_2} + a_2\nu_{\ell_2} - a_2\nu_{\ell_1})Y_2 = \lambda - \nu_{\ell_1} D.$$

Using (3.15) and (3.16) it follows that

$$a_1(\nu_i - \nu_{\ell_1})Y_1 + a_2(\nu_i - \nu_{\ell_1})Y_2 = \lambda - \nu_{\ell_1} D.$$

Noticing that $(Y_1, Y_2) \in \mathcal{Y}_1$ and by (iii) $\nu_i - \nu_{\ell_1} < 0$, then $\lambda - \nu_{\ell_1} D = (\nu_i - \nu_{\ell_1}) \min\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \mathcal{Y}_1\} = (\nu_i - \nu_{\ell_1})(D + \epsilon_1)$, where $\epsilon_1 = \min\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \mathcal{Y}_1\} - D$. Similarly, using a solution in $\mathcal{F}_{(\mu,\nu,\lambda)}$ satisfying $y_i = 0$ and $0 < X_2 < a_2 Y_2$ we conclude $\lambda - \nu_{\ell_2} D = (\nu_t - \nu_{\ell_2})(D + \epsilon_2)$ where $\epsilon_1 = \min\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \mathcal{Y}_2\} - D$.

Next we show that

$$\mu_j = \mu_i \text{ and } \nu_j = \nu_i, \forall j \in I \tag{3.17}$$

Suppose conditions (3.17) do not hold for some $j \in I_1$ (for $j \in I_2$ the proof is similar). For each point in $\mathcal{F}_{(\mu,\nu,\lambda)}$ with $x_j > 0$ we have

$$(\mu_{\ell_1} + a_1 \nu_{\ell_1})Y_1 + (\mu_{\ell_2} + a_2 \nu_{\ell_2})Y_2 + \mu_j + \nu_j x_j = \lambda$$

$$\Leftrightarrow a_1 \nu_i Y_1 + a_2 \nu_i Y_2 + \mu_j + \nu_j (D - a_1 Y_1 - a_2 Y_2) = \lambda$$

$$\Leftrightarrow a_1(\nu_i - \nu_j)Y_1 + a_2(\nu_i - \nu_j)Y_2 = \lambda - \nu_j D - \mu_j.$$

Thus $(Y_1, Y_2) = argmax\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \Sigma^1\}$ if $\nu_i > \nu_j$ and $(Y_1, Y_2) = argmin\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \Sigma^1\}$, otherwise. Let us define $\varphi = D - \max\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \Sigma^1\}$ if $\nu_t > \nu_j$ and $\varphi = D - min\{a_1 Y_1' + a_2 Y_2' : (Y_1', Y_2') \in \Sigma^1\}$ otherwise. Hence $x_j = \varphi y_j$ for each point in $\mathcal{F}_{(\mu,\nu,\lambda)}$, which is a contradiction.

Using (3.17) and the six equations: (3.15); (3.16); $\lambda = D\nu_i + \mu_i$; $\lambda = \nu_i(D - \gamma)$; $\lambda - \nu_{\ell_1} D = (\nu_i - \nu_{\ell_1})(D + \epsilon_1)$; $\lambda - \nu_{\ell_2} D = (\nu_i - \nu_{\ell_2})(D + \epsilon_2)$; we conclude that the facet is defined by inequality (3.6).

Case 2. $\dfrac{\mu_{\ell_2} + a_2 \nu_{\ell_2}}{\mu_{\ell_1} + a_1 \nu_{\ell_1}} \neq \dfrac{a_2}{a_1}$. Next we show that the inequality defined by $(\mu_{\ell_1}, \mu_{\ell_2}, \nu_{\ell_1}, \nu_{\ell_2}, \lambda)$:

$$\mu_1 \sum_{j \in S_1} y_j + \mu_2 \sum_{j \in S_2} y_j + \nu_1 \sum_{j \in S_1} x_j + \nu_2 \sum_{j \in S_2} x_j \leq \lambda \tag{3.18}$$

must define a facet of the polyhedron with $N' = N \setminus I$. Since $\mathcal{F}_{(\mu,\nu,\lambda)}$ is a facet of P it includes a set $\mathcal{A}$ with $2n$ affinely independent points. For each $i \in I$ there must exist in $\mathcal{A}$ at least two points with $y_i = 1$. However, since each point in $\mathcal{F}_{(\mu,\nu,\lambda)} \in X$ with $y_i = 1$, $i \in I_1$, corresponds to an optimal solution to $AP1$, it follows that there are in $\mathcal{A}$ exactly two points with $y_i = 1$ for each $i \in I_1$ (observe that the hypothesis $\frac{\mu_{\ell_2} + a_2 \nu_{\ell_2}}{\mu_{\ell_1} + a_1 \nu_{\ell_1}} \neq \frac{a_2}{a_1}$ implies that the coefficients of the objective function of AP1 are not simultaneously null). Thus $\mathcal{A}$ includes $2(n- \mid I_1 \mid)$ affinely independent points with $x_i = y_i = 0$ for $i \in I_1$. Similarly, considering $i \in I_2$, we conclude that $\mathcal{A}$ includes $2(n- \mid I_2 \mid)$ affinely independent points with $x_i = y_i = 0$ for $i \in I_2$. This implies that (3.18) defines a facet of the polyhedron with $N' = N \setminus I$. From Lemma 3.1, we conclude that it defines a facet for the 2-integer variables model Z.

As (3.14) is valid, then $(\mu_i, \nu_i) \in \Theta^1_{(\mu_u,\mu_\ell,\nu_u,\nu_\ell,\lambda)}$. Observe that the points in $\mathcal{F}_{(\mu,\nu,\lambda)} \cap X$ with $y_i = 1$ are obtained from two optimal solutions to $AP1$ which belong to $V(\Sigma^1)$. These two extreme points in $V(\Sigma^1)$ give the two tight constraints of $\Theta^1_{(\mu_u,\mu_\ell,\nu_u,\nu_\ell,\lambda)}$ defining the extreme point (μ_i, ν_i).

Hence, (3.14) is of type (3.13). □

3.4. Separation

In this section we study the separation problems associated with the families of valid inequalities derived for X. Consider a point $(y, x) \in \mathbb{R}^{2n}$. For each family of valid inequalities the separation problem is: find an inequality that is violated by (y, x) or show that no such inequality exists.

First we consider inequalities (3.6). Sets S_1, S_2 maximize the left-hand side of (3.6) if and only if:

$$\{j \in N_1 : x_j - \gamma y_j < a_1 y_j + \frac{\gamma+\epsilon_1}{\epsilon_1} x_j - a_1 \frac{\gamma+\epsilon_1}{\epsilon_1} y_j\} \subseteq S_1,$$

$$\{j \in N_1 : x_j - \gamma y_j > a_1 y_j + \frac{\gamma+\epsilon_1}{\epsilon_1} x_j - a_1 \frac{\gamma+\epsilon_1}{\epsilon_1} y_j\} \subseteq N \setminus S_1,$$

$$\{j \in N_2 : x_j - \gamma y_j < a_2 y_j + \frac{\gamma+\epsilon_2}{\epsilon_2} x_j - a_2 \frac{\gamma+\epsilon_2}{\epsilon_2} y_j\} \subseteq S_2,$$

$$\{j \in N_2 : x_j - \gamma y_j > a_2 y_j + \frac{\gamma+\epsilon_2}{\epsilon_2} x_j - a_2 \frac{\gamma+\epsilon_2}{\epsilon_2} y_j\} \subseteq N \setminus S_2.$$

For those cases where the equality $x_j - \gamma y_j = a_t y_j + \frac{\gamma+\epsilon_t}{\epsilon_t} x_j - a_t \frac{\gamma+\epsilon_t}{\epsilon_t} y_j$ holds, for $t \in \{1, 2\}$, choose arbitrarily between the corresponding sets S_t, $N \setminus S_t$. The separation problem can be solved in $\mathcal{O}(n)$. Next we describe a simple separation procedure for the lifted Euclidean inequalities.

For each Euclidean inequality of type (2.1), (2.2), (2.3) we construct a lifted Euclidean inequality (3.13). In order to maximize the left-hand side we choose the set in which to put $j \in N$ as follows:

For $j \in N_1$ we determine $\max\{0, \mu_1 y_j + \nu_1 x_j, \theta_1^k y_j + \xi_1^k x_j\}$ where $k = argmax_{\ell \in T_1}\{\theta_1^\ell y_j + \xi_1^\ell x_j\}$. Then put j in $N_1 \setminus \{S_1 \cup I_1\}$ if the maximum is 0; put j in S_1 if the maximum is $\mu_1 y_j + \nu_1 x_j$; and put $j \in I_1^k$ otherwise.

For $j \in N_2$ we determine $\max\{0, \mu_2 y_j + \nu_2 x_j, \theta_2^k y_j + \xi_2^k x_j\}$ where $k = argmax_{\ell \in T_2}\{\theta_2^\ell y_j + \xi_2^\ell x_j\}$. Then put j in $N_2 \setminus \{S_2 \cup I_2\}$ if the maximum is 0; put j in S_2 if the maximum is $\mu_2 y_j + \nu_2 x_j$; and put $j \in I_2^k$ otherwise.

The number of different coefficients in I is $\mathcal{O}(b)$ where $b = log(D/\min\{a_1, a_2\})\}$ and the time complexity to compute those coefficients is similar. The overall procedure is $\mathcal{O}(nb)$

4. Applications and Computational experience

In this section we discuss three possible applications: an inventory-distribution problem, a capacitated facility location problem, and a muti-item production planning problem. The main purpose is to illustrate and explain how the lifted Euclidean inequalities can be used in practical problems. We report on computational experimentations on small sets of instances for each one of these three problems. With the first two problems we also illustrate how these inequalities can be used in sets that are very similar to set $\mathcal{X}$ but do not coincide with $\mathcal{X}$.

For each problem we compare the value of the linear relaxation with the value of the linear relaxation after the inclusion of cuts from the lifted Euclidean inequalities. In order to derive these cuts, we consider a single node flow set obtained by relaxation of the original feasible set. Then we add all valid inequalities derived for the corresponding single node flow set that are violated by the fractional linear programming solution. We repeat this process until no further violated valid inequalities are obtained.

For the computation we use the optimization package Xpress Optimizer, Version 25.01.05, with MOSEL in a computer with a Intel Core I7, 2.4GHz processor with 16GB RAM.

4.1. Vendor Management Problem

The Vendor Management Problem (VMP) occurs when a distributor/producer controls the inventory at the retailers. Given a set of n retailers and a demand in each period of each retailer during a time horizon of m periods, the VMP aims to find the amount to order in each time period of a given item and the amount to send to each retailer in each period, in order to minimize the holding, backlogging, distribution and fixed ordering costs.

Define $T = \{1, \ldots, m\}$ and $I = \{1, \ldots, n\}$ the set of time periods and retailers, respectively. We assume there is a fleet with two types of vehicles with capacities C_1 and C_2, and each retailer is visited by one type of vehicle only. Let I_1, I_2 define the subsets of I that are served by vehicles with capacity C_1, C_2, respectively.

For each period $t \in T$ and each retailer $i \in I$, consider the variables x_i^t, y_i^t, s_i^t that represent, respectively, the amount sent, in time period t, to client i; the number of vehicles used to serve retailer i, in period t; and the stock level in retailer i at the end of time period t. The binary variable z_t indicates whether a fixed cost is incurred in period t or not.

For each time period t and each retailer i, d_i^t represents the demand, p_i^t, represents the unit product transportation cost, f_i^t represents the fixed transportation cost per vehicle, and h_i^t represents the unit product holding cost. M is distribution capacity in each time period and g_t is the fixed cost for distributing in period t. The VMP can be written as follows.

$$
\begin{aligned}
min \quad & \textstyle\sum_{i\in I}\sum_{t\in T} p_i^t x_i^t + f_i^t y_i^t + h_i^t s_i^t + \sum_{t\in T} g_t z_t && \\
& s_i^{t-1} + x_i^t = d_i^t + s_i^t, && i \in I, t \in T, && (4.1)\\
& x_i^t \leq C_1 y_i^t, && i \in I_1, t \in T, && (4.2)\\
& x_i^t \leq C_2 y_i^t, && i \in I_2, t \in T, && (4.3)\\
& \textstyle\sum_{i\in I} x_i^t \leq M z^t && t \in T, && (4.4)\\
& x_i^t, s_i^t \geq 0, y_i^t \in \{0,1\} && i \in I, t \in T, && (4.5)\\
& s_i^0 = s_i^m = 0 && i \in I, && (4.6)\\
& z_t \in \{0,1\} && t \in T. && (4.7)
\end{aligned}
$$

Constraints (4.1) are the usual flow conservation constraints at each retailer and for each time period. Constraints (4.2) and (4.3) are the variable upper bound constraints. Constraints (4.4) impose a maximum amount to distribute in each period and impose a setup cost whenever there is distribution. Constraints (4.5) - (4.7) are the sign constraints. The set defined by (4.1) - (4.7) will be denoted by X^{VMP}.

Consider the relaxation of the VMP obtained by eliminating the flow conservation constraints. The relaxed model is separable into several subproblems, one for each period t. For each period t, the feasible set, denoted by X_t^{VMP}, is the integer single node flow set with two capacities and with a set-up variable associated to the node itself:

$$X_t^{VMP} = \{(z, y, x) \in \{0,1\} \times \mathbb{Z}_+^n \times \mathbb{R}^n : \sum_{i\in I} x_i \leq Mz, 0 \leq x_i \leq C_1 y_i, i \in I_1, 0 \leq x_i \leq C_2 y_i, i \in I_2\}.$$

Set X^{VMP} differs from X (considering I as N and C_1, C_2 as a_1, a_2) since it includes the set-up variable z. Following Proposition 4 in [3], one can easily show that if

$$\sum_{j\in I} \alpha_j x_j + \sum_{j\in I} \beta_j y_j \leq \delta \tag{4.8}$$

is valid for X, then (4.8) is valid for X^{VMP}. Conversely, if $\beta_j \leq 0, \forall j \in N$, then inequality (4.8) is valid for X, if and only if

$$\sum_{j\in I} \alpha_j x_j + \sum_{j\in I} \beta_j y_j \leq \delta z \tag{4.9}$$

is valid for X_t^{VMP}. Hence all the inequalities derived for X can be used directly to tighten the linear relaxation of X^{VMP}, and tighter inequalities can be derived by multiplying the RHS of inequalities (3.6) and (3.13) by z_t.

Such problem occurs, for instance, within maritime transportation, in a medium-term planning, when a product is supplied by large ships (batches in our model), and then distributed among a set of ports using an heterogeneous fleet (with two types) of smaller ships.

Each port is served by one type of ship (accordingly to the characteristics of each port). The set-up variable z^t indicates whether a large shipment must occur in that time period and the y variables indicate the number of smaller ships of type t that must be sent to a given port.

We consider instances with $m = 20$, $n = 5$. Demands d_i^t were randomly generated in $[1, 25]$, and we set the unit transportation costs to $p_i^t = 2$, the fixed transportation costs to $f_i^t = 1001$, the unit holding costs $h_i^t = 0$. $M = 1000$. We consider 5 instances with $C_1 = 10, C_2 = 17$, labeled as $i1a, \dots, i5a$ and 5 instances with capacities $C_1 = 30, C_2 = 50$, which are labeled as $i1b, \dots, i5b$.

The computational results after 5 minutes of computer time are reported in Table 1. Column LR indicates the linear relaxation value, BFB gives the Best known Feasible Solution, Gap gives the corresponding gap ($Gap = \frac{BFB-LR}{BFB} * 100\%$). Columns $LR + C$ and $Gap + C$ give the linear relaxation value and the corresponding gap after the cuts have been added at the root node. Column Cuts gives the number of cuts added. The last columns give the Best Feasible Solution (BFS), the Best Lower Bound (BLB), which is the best lower bound obtained at the end of the running time, and the corresponding gap (Gap) for the cases with inclusion of cuts and without inclusion of cuts.

We can see that the inclusion of cuts reduced both the initial gap and the gap after 5 minutes of running time.

4.2. Capacitated facility location problem

Consider the following capacitated facility location problem (CFLP). We are given a set $N = \{1, \dots, n\}$ of clients and a set $L = \{1, \dots, m\}$ of possible facility locations. Let d_t represent the demand of client t, and C_j represent the capacity of each facility $j \in L$. Parameter f_j indicates the fixed cost for installing a facility in $j \in L$ and p_{jt} indicates the cost of satisfying one unit of demand of client $t \in T$ from a facility $j \in N$. The variables x_{jt} indicate the amount of the demand of client t, that is satisfied by the facilities located at j. The integer variables y_j represent the number of facilities open at location j. The CFLP can be described as follows.

$$
\begin{aligned}
min \quad & \textstyle\sum_{t\in N} \sum_{j\in L} p_{jt}x_{jt} + \sum_{j\in L} f_j y_j \\
& \textstyle\sum_{j\in L} x_{jt} = d_t, && t \in N, && (4.10)\\
& \textstyle\sum_{t\in N} x_{jt} \le C_j y_j, && j \in L, && (4.11)\\
& x_{jt} \le d_t y_j, && j \in L, t \in N, && (4.12)\\
& x_{jt} \ge 0, && j \in L, t \in N, && (4.13)\\
& y_j \in \mathbb{Z}_0^+, && j \in L. && (4.14)
\end{aligned}
$$

Constraints (4.10) ensure that the demand of each client is satisfied. Constraints (4.11) impose the capacity of each facility. Constraints (4.12) are the usual variable upper bound constraints, used to tighten the formulation .

Table 1: Computational tests for a set of vendor management instances with and without cuts.

						With Cuts				Without Cuts		
	LR	**BFS**	**Gap**	**LR+C**	**Gap+C**	**Cuts**	**BFS**	**BLB**	**Gap**	**BFS**	**BLB**	**Gap**
i1a	115591	118782	2.7	115796	2.5	101	118782	117181	1.3	118782	115591.2	2.7
i1b	49580.1	52716	5.9	50135.7	4.9	62	52716	51248	2.8	52716	49580.13	5.9
i2a	104459	107530	2.9	104664	2.7	116	107530	105518.8	1.9	107530	105268.9	2.1
i2b	44878.2	48471	7.4	45364.9	6.4	57	48471	46421.84	4.2	48471	46669.2	3.7
i3a	107793	110541	2.5	107977	2.3	113	110541	108970.2	1.4	110541	110363.4	0.2
i3b	45994	49480	7.0	46562.8	5.9	60	49480	47341.55	4.3	49480	47744.9	3.5
i4a	114772	116722	1.7	114989	1.5	115	116722	115832	0.8	116722	116545.4	0.2
i4b	49052	51657	5.0	49640.8	3.9	61	51657	49640.8	3.9	51657	50942.0	1.4
i5a	114833	118792	3.3	115077	3.1	116	118792	117441.2	1.1	118792	116283.6	2.1
i5b	49379.9	52726	6.3	49905.4	5.3	62	52726	50934.4	3.4	52726	49379.9	6.3
Av.			4.5		3.9				2.5			2.8

Considering a subset $T \subseteq N$ of clients, and aggregating the corresponding variables $w_j = \sum_{t\in T} x_{jt}$ we obtain the following single node flow set:

$$X^{FC} = \{(y,w) \in \mathbb{Z}^n \times \mathbb{R}^n_+ : \sum_{j\in L} w_j = \sum_{t\in T} d_t,\ w_j \leq C_j y_j,\ j \in L\}.$$

Assume there are only two possible values for capacity C_j. Set X^{FC} differs from X because it has the equality constraint $\sum_{j\in L} w_j = \sum_{t\in T} d_t$. Obviously, valid inequalities for set X (with $\leq$ constraints) are valid for X^{FC} since X^{FC} is a restriction of X. For the equality case the families (3.6) are void (since it assumes $\gamma > 0$, $\epsilon_1 > 0$, and $\epsilon_2 > 0$). Similarly, in family (3.8) we have $I = \emptyset$.

We consider instances with $n = 100$, $m = 10$. Parameters d_t, f_j were randomly generated in $[100, 150]$, and $[9000, 10000]$, respectively. And $p_{jt} = 2, \forall j, t$. Again we consider 5 instances with $C_1 = 100, C_2 = 170$, labeled as $i1a, \ldots, i5a$ and 5 instances with $C_1 = 300, C_2 = 500$, labeled as $i1b, \ldots, i5b$.

The computational results are reported in Table 2. Since all the instances were solved to optimality, in addition to the notation introduced before, column OPT indicates the value of the optimal solution, columns $Time$ indicate the running time in seconds to solve the problem and columns $Nodes$ indicate the number of nodes in the branch-and-bound tree.

Table 2: Computational tests for a set of facility location instances with and without cuts.

				With Cuts					Without Cuts	
	LR	**OPT**	**Gap**	**LR+C**	**Gap+C**	**Cuts**	**Nodes**	**Time**	**Nodes**	**Time**
i1a	712659	719997	1.0	717446	0.4	3	1	0	64509	196
i1b	258732	259765	0.4	259765	0.0	2	1	0	63	0
i2a	706786	708416	0.2	708416	0.0	3	1	0	19	0
i2b	256873	265184	3.1	265184	0.0	2	1	0	61	0
i3a	744514	744800	0.04	744800	0.0	2	1	0	16	0
i3b	269734	276552	2.5	276462	0.0	2	1	0	1675	1
i4a	703383	704550	0.2	704550	0.0	3	1	0	75	0
i4b	255279	260162	1.9	257727	0.9	1	347	0	3	0
i5a	701599	703378	0.3	703378	0.0	3	1	0	13	0
i5b	255106	263124	3.0	263086	0.0	2	1	0	1	0

The inclusion of cuts was again effective in reducing the integrality gap and the number of branch-and-bound nodes, in average.

4.3. A muti-item production planning problem

Consider the following muti-item production planning problem (MPP). Let $I = \{1, \ldots, n\}$ be the set of items and $T = \{1, \ldots, m\}$ be the set of time periods. For each period $t \in T$

consider the variables x_i^t, y_i^t, s_i^t and r_i^t, that represent the production lot sizing of item i in period t; the integer variable indicating the number batches of item i to produce in period t; the inventory of item i at the end of period t; and the backlog of item i at the end of period t, respectively.

The demand of each item for each period is given by d_i^t. For each time period t, M^t is the available production capacity. In each time period we consider two batch sizes. One with capacity c_1^t for items $i \in I_1$ and the other, c_2^t, for items $i \in I_2 = I \setminus I_1$. For each item and each period the costs $p_i^t, f_i^t, h_i^t, b_i^t$ represent, respectively, the unit production cost, the fixed production cost, the unit inventory cost and the unit backlog cost.

The multi-item lot-sizing problem is given by,

$$
\begin{aligned}
min \quad & \textstyle\sum_{i\in I}\sum_{t\in T} p_i^t x_i^t + f_i^t y_i^t + h_i^t s_i^t + b_i^t r_i^t \\
& s_i^{t-1} + x_i^t + r_i^t = d_i^t + s_i^t + r_i^{t-1}, && i \in I, t \in T, && (4.15)\\
& s_i^0 = s_i^m = r_i^0 = r_i^m = 0, && i \in I, && (4.16)\\
& x_i^t \leq c_1^t y_i^t, && i \in I_1, t \in T, && (4.17)\\
& x_i^t \leq c_2^t y_i^t, && i \in I_2, t \in T, && (4.18)\\
& \textstyle\sum_{i\in I} x_i^t \leq M^t, t \in T, && && (4.19)\\
& x_i^t, s_i^t \geq 0, y_i^t \in \{0,1\}, && i \in I, t \in T. && (4.20)
\end{aligned}
$$

Constraints (4.15) are the flow conservation constraints. Constraints (4.16) ensure that the inventory and backlogging at the beginning and at the end of the planning horizon is zero for each item. Constraints (4.17) and (4.18) establish the upper bound capacity on each lot. Constraints (4.19) model a resource constraint (as time-machine constraint).

Consider the relaxation of the MPP obtained by deleting the flow conservation constraints. The relaxed model is separable in several problems, one for each period t. For each period t, the feasible set is the integer single node flow set with two possible values for the upper bounds:

$$\{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^n : \sum_{i\in I} x_i \leq M, 0 \leq x_i \leq c_1 y_i, i \in I_1, 0 \leq x_i \leq c_2 y_i, i \in I_2\}.$$

We consider instances $n = 5$, $m = 20$, $p_i^t = 2, f_i^t = 101, h_i^t = 1, \forall i, t$. Parameters d_t, b_i^t, were randomly generated in intervals $[1, 25]$, and $[500, 600]$, respectively. Again we consider 5 instances with $C_1 = 10, C_2 = 17$, labeled as $i1a, \ldots, i5a$ and 5 instances with $C_1 = 30, C_2 = 50$, labeled as $i1b, \ldots, i5b$.

The computational results are reported in Table 3. The meaning of the columns is the same as for the VMP.

Table 3: Computational tests for a set of multi-item lot-sizing instances with and without cuts.

				With Cuts						Without Cuts		
	LR	**BFS**	**Gap**	**LR+C**	**Gap+C**	**Cuts**	**BFS**	**BLB**	**Gap**	**BFS**	**BLB**	**Gap**
i1a	13314.4	13820	3.7	13392	3.1	30	13817	13497.7	2.3	13820	13466.3	2.6
i1b	6365.5	7338	13.3	6883.2	6.2	103	7346	7217.7	1.7	7338	7224.2	1.6
i2a	13226.5	13313.2	0.7	13290.6	0.2	42	13453	13313.2	1.0	13445	13301	1.1
i2b	6293.5	6956	9.5	6760	2.8	123	6956	6916.7	0.6	6957	6917	0.6
i3a	12591.4	13113	4.0	12957.8	1.2	90	13139	12972.4	1.3	13113	12881.9	1.8
i3b	6429.26	7445	13.6	7364.4	1.1	94	7446	7442	0.1	7445	7445	0.0
i4a	12789.3	13363	4.3	12957.8	3.0	47	13363	13039.3	2.4	13386	12992.3	2.9
i4b	6121.3	7132	14.2	6758.13	5.2	94	7132	7053	1.1	7102	7064.1	0.5
i5a	13347.5	13783	3.2	13521	1.9	84	13808	13549.2	1.8	13783	13497.4	2.1
i5b	6775	7591	10.7	7473.68	1.5	111	7591	7584.4	0.1	7591	7587.5	0.0
Av.			7.7		2.6				1.2			1.3

As for the previous two models, the value of linear programming gaps have improved in the presence of the lifted Euclidean inequalities, while the final gaps were slightly better for most instances.

5. Appendix

Proof: (Proposition 3.3) To ease the notation of this proof, for set S we denote by $X(S)$ and $Y(S)$ the sums $\sum_{j\in S} x_j$ and $\sum_{j\in S} y_j$, respectively.

Consider a point $(y,x)\in\mathcal{X}$. Case 1. $Y(N\setminus(S_1\cup S_2))=0$. If $(Y(S_1),Y(S_2))\in\mathcal{Y}_{\leq}$ then

$$a_1Y(S_1)+a_2Y(S_2)+\frac{\gamma+\epsilon_1}{\epsilon_1}(X(S_1)-a_1Y(S_1))+\frac{\gamma+\epsilon_2}{\epsilon_2}(X(S_2)-a_2Y(S_2))+(X(S_1\cup S_2)-\gamma Y(S_1\cup S_2))$$

$$\leq a_1Y(S_1)+a_2Y(S_2)\leq D-\gamma$$

Now suppose $(Y(S_1),Y(S_2))\in\mathcal{Y}_{\geq}=\{(y_1,y_2)\in\mathbb{Z}^2_+ : a_1y_1+a_2y_2\geq D\}$. W.l.o.g. suppose $(Y(S_1),Y(S_2))\in\mathcal{Y}_1$. Since it suffices to prove validity for the extreme points of $\mathcal{P}$, we may assume $X(S_1)=D-a_2Y(S_2)$ and $X(S_2)=a_2Y(S_2)$. Thus, as $\epsilon_1\leq\min\{a_1y_1+a_2y_2-D : (Y_1,Y_2)\in\mathcal{Y}_1)\}$, then,

$$\epsilon_1\leq(a_1Y(S_1)+a_2Y(S_2)-D)$$

$$\Rightarrow \gamma(D-a_1Y(S_1)-a_2Y(S_2))+\epsilon_1(D-a_1Y(S_1)-a_2Y(S_2))\leq\epsilon_1(D-a_1Y(S_1)-a_2Y(S_2))-\epsilon_1\gamma$$

$$\Rightarrow (\gamma+\epsilon_1)(D-a_1Y(S_1)-a_2Y(S_2))\leq\epsilon_1(D-\gamma-a_1Y(S_1)-a_2Y(S_2))$$

$$\Rightarrow (\gamma+\epsilon_1)(X(S_1)-a_1Y(S_1))\leq\epsilon_1(D-\gamma-a_1Y(S_1)-a_2Y(S_2))$$

$$\Rightarrow a_1Y(S_1)+a_2Y(S_2)+\frac{\gamma+\epsilon_1}{\epsilon_1}(X(S_1)-a_1Y(S_1))\leq D-\gamma$$

$$\Rightarrow a_1Y(S_1)+a_2Y(S_2)+\frac{\gamma+\epsilon_1}{\epsilon_1}(X(S_1)-a_1Y(S_1))$$

$$+\frac{\gamma+\epsilon_2}{\epsilon_2}(X(S_2)-a_2Y(S_2))+X(N\setminus(S_1\cup S_2))-\gamma Y(N\setminus(S_1\cup S_2))\leq D-\gamma$$

Case 2. $Y(N\setminus(S_1\cup S_2))>0$.

$$a_1Y(S_1)+a_2Y(S_2)+\frac{\gamma+\epsilon_1}{\epsilon_1}(X(S_1)-a_1Y(S_1))$$

$$+\frac{\gamma+\epsilon_2}{\epsilon_2}(X(S_2)-a_2Y(S_2))+X(N\setminus(S_1\cup S_2))-\gamma Y(N\setminus(S_1\cup S_2))$$

$$\leq\frac{\gamma+\epsilon_1}{\epsilon_1}\left(X(S_1)-\frac{\gamma}{\gamma+\epsilon_1}a_1X(S_1)\right)+\frac{\gamma+\epsilon_2}{\epsilon_2}\left(X(S_2)-\frac{\gamma}{\gamma+\epsilon_2}a_2Y(S_2)\right)+X(N\setminus(S_1\cup S_2))-\gamma$$

$$\leq\frac{\gamma+\epsilon_1}{\epsilon_1}\left(X(S_1)-\frac{\gamma}{\gamma+\epsilon_1}X(S_1)\right)$$

$$+\frac{\gamma+\epsilon_2}{\epsilon_2}\left(X(S_2)-\frac{\gamma}{\gamma+\epsilon_2}X(S_2)\right)+X(N\setminus(S_1\cup S_2))-\gamma$$

$$\leq X(S_1) + X(S_2) + X(N \setminus (S_1 \cup S_2)) - \gamma \leq D - \gamma$$

$\square$

Proof: (Lemma 3.4)

First observe that (a) $\mu_t < 0$, and (b) $\nu_t > 0$, for $t \in \{1,2\}$ (see [1]). Next we show that if (3.8) defines a facet of P with $(\mu_1, \mu_2, \nu_1, \nu_2) \in \Lambda$, and $(\mu_1, \nu_1) \neq (\theta_j, \eta_j), j \in I_1$, $(\mu_2, \nu_2) \neq (\theta_j, \eta_j), j \in I_2$, then following conditions hold:

(c) $\theta_j + a_t\eta_j < \mu_t + a_t\nu_t,\ t \in \{1,2\}, j \in I_t$;

(d) $\eta_j < \nu_t,\ t \in \{1,2\}, j \in I_t$;

(e) $\theta_j > \mu_t,\ t \in \{1,2\}, j \in I_t$.

We give the proof of (c) - (e) for case $t = 1$, since the other one is similar. Suppose $\theta_j + a_1\eta_j > \mu_1 + a_1\nu_1, j \in I_1$. As $(\mu_1, \mu_2, \nu_1, \nu_2) \in \Lambda$ and (3.8) defines a facet $\mathcal{F}$ for the restricted set, then there must exist a tight point in $\mathcal{F}$ with $Y(S_1) \geq 1$ and $X(S_1) \geq a_1$. Thus, a new point (x', y') violating (3.8) can be created by setting $y'_j = 1, x'_j = a_1, Y'(S_1) = Y(S_1) - 1, X'(S_1) = X(S_1) - a_1$ and keeping the remaining variables unchanged. Hence, $\theta_j + a_1\eta_j \leq \mu_1 + a_1\nu_1$. If $\theta_j + a_1\eta_j = \mu_1 + a_1\nu_1$ and $(\mu_1, \nu_1) \neq (\theta_j, \eta_j)$, then either (A) $\theta_j < \mu_1$ and $\eta_j > \nu_1$ or (B) $\theta_j > \mu_1$ and $\eta_j < \nu_1$. In case (A) we have $\theta_j + \gamma\eta_j < \mu_1 + \gamma\nu_1$ for all $0 < \gamma < a_1$ (since $\nu_1 \geq 0$) which implies $x_j = a_1y_j$ for all points in $\mathcal{F}$. Similarly, in case (B) we have $x_1 = a_1y_1$ for all points in $\mathcal{F}$. Hence (c) holds.

As (c) holds, it must occur, for each $t \in \{1,2\}$ and $j \in I_t$, $\theta_j + \gamma\eta_j \geq \mu_t + \gamma\nu_t$, for some $0 < \gamma < a_t$, otherwise we would have $x_j = y_j = 0$ for all tight points in the facet. Together with (c) this implies (d) and (e).

Now, we resume the proof of Lemma 3.4. Again we consider case $t = 1$. Suppose (x, y) is a tight extreme point of $\mathcal{P}$ in the face defined by (3.8) and $y_k > 1$, for some $k \in I_1$. If $x_k \leq a_1(y_k - 1)$ we can decrease the value of y_k by one unit and obtain a feasible solution violating (3.8) because, by (a), $\theta_k < 0$. So consider the case $a_1(y_k - 1) < x_k$. Then decreasing the value of y_k by one and decreasing the value of x_k by a_1 and increasing the value of y_i and x_i by the same amounts we obtain a feasible solution violating (3.8). This proves (i).

The case (ii) follows from the fact that for network flow sets every extreme point of $\mathcal{P}$ can have at most one continuous variable $x_j, j \in N$ with value $0 < x_j < a_jy_j$. So, as $y_k = 1$ then if $y_j > 0, j \in I \setminus \{k\}$ either $x_j = 0$ or $x_j = a_ty_j$. We can construct a violated feasible point using (a) in case $x_j = 0$ (decreasing y_j), and using (c) in case $x_j = a_ty_j$ (transferring the amounts in x_j and y_j to some x_i, and y_i with $i \in S_t$.

Properties (iii) and (iv) follow again from the same arguments and noticing that in such extreme point with $y_k = 1, k \in I$, then $\sum_{j \in N} x_j = D$. $\square$

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